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Monte Carlo simulation for radiative kaon decays

C. Gatti[a]

Laboratori Nazionali di Frascati – INFN Via Enrico Fermi 40, 00044 Frascati (Rome), Italy

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Abstract. For high precision measurements of K decays, the presence of radiated photons cannot be neglected. The Monte Carlo simulations must include the radiative corrections in order to compute the correct event counting and for efficiency calculations. In this paper, a method for simulating such decays is briefly described.

1 Introduction

Many measurements on K decays have reached a statistical error close to 1% or less. With such precision, the presence of radiated photons, and in general the effect of radiative corrections, cannot be neglected. Furthermore, the treatment of the radiative corrections is explicitly required for the extraction of many physical quantities, such as the CKM matrix element V_{us} and the phase shifts $\delta_0 - \delta_2$, at precisions of a few percent (or above). Hence, it is mandatory to include the effect of radiated photons in the Monte Carlo (MC) simulations.

There are two aspects of a measurement that are affected by the presence of radiated photons [1]: the geometrical acceptance and the counting of the events. About 2.4% of $K^0 \to \pi e \nu \gamma$ decays have a photon with an energy above 30 MeV, as shown in Fig. 1, corresponding to about 10% of the center-of-mass energy. These photons soften the momentum spectra of the electron and pion that are actually detected in an experiment, changing the geometrical acceptance and the distributions of kinematic quantities often used to select or count the number of signal events. If these effects are neglected, errors of up to few percent can affect the measurement of a given branching ratio [2, 3].

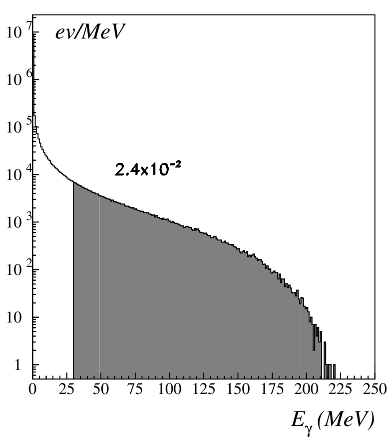

Fig. 1. Energy spectrum for $K^0 \to \pi e \nu \gamma$ MC decays. 2.4% of the events have $E_\gamma > 30\,\mathrm{MeV}$

2 Bremsstrahlung and infrared divergences

The main problem in simulating radiative decays is the presence of infrared divergences: the total decay width for single photon emission, computed at any fixed order in α, is infinite. A finite value is obtained only by summing the decay widths for the real and virtual processes calculated to the same order in α [4–6]. As shown in [6], in the limit of soft-photon energy, we can "re-sum" the probabilities for multiple photon emission to all orders in α. The rate for the decay process $i \to f$ accompanied by any number of soft photons with total energy less than E_γ is given by

$$\Gamma_{\mathrm{incl}}(E_\gamma) = \Gamma_0 \left(\frac{E_\gamma}{\Lambda}\right)^b (1 + O(b^2) + O(E_\gamma)). \qquad (1)$$

Here Γ_0 is the unphysical decay width for the process $i \to f$ without final state photons, and b is a function of the particle momenta, is positive and of order α; it is given by

$$b = -\frac{1}{8\pi^2} \sum_{m,n} \eta_m \eta_n e_m e_n \beta_{mn}^{-1} \ln \frac{1+\beta_{mn}}{1-\beta_{mn}}, \qquad (2)$$

where m and n run over all the external particles, e_n is the charge of particle n, $\eta = +1$ or -1 for an outgoing or incoming particle, and β_{mn} is the relative velocity of the particles n and m in the rest frame of either:

$$\beta_{mn} = \left[1 - \frac{m_n^2 m_m^2}{(p_n \cdot p_m)^2}\right]^{1/2}. \qquad (3)$$

Λ is an energy cut-off that can be chosen as the mass M of the decaying particle.

Differentiating $\Gamma_{\mathrm{incl}}(E_\gamma)$ with respect to E_γ, we obtain an integrable differential distribution:

$$\frac{\mathrm{d}\Gamma_{\mathrm{incl}}}{\mathrm{d}E_\gamma} = \Gamma_0 b \frac{E_\gamma^{b-1}}{M^b} = \frac{\mathrm{d}\Gamma_{\mathrm{Brem}}}{\mathrm{d}E_\gamma} \left(\frac{E_\gamma}{M}\right)^b, \qquad (4)$$

[a] e-mail: claudio.gatti@lnf.infn.it

where we have neglected second order terms in b. To $O(\alpha)$ this can be identified with the single-photon emission probability, and indeed, it can be written in terms of

$$\frac{\mathrm{d}\Gamma_{\mathrm{Brem}}}{\mathrm{d}E_\gamma} = \Gamma_0 \frac{b}{E_\gamma}. \tag{5}$$

The presence of the extra factor $(E_\gamma/M)^b$ ensures the integrability of (4) in the limit $E_\gamma \to 0$.

In the derivation of (1), in [6], no explicit integration is required on the momenta of particles other than those of the photons. The result in (4) can thus be applied to differential decay widths:

$$\frac{\mathrm{d}\Gamma_{\mathrm{incl}}}{\mathrm{d}E_\gamma \mathrm{d}\underline{\xi}} = \frac{\mathrm{d}\Gamma_{\mathrm{Brem}}}{\mathrm{d}E_\gamma \mathrm{d}\underline{\xi}} \left(\frac{E_\gamma}{M}\right)^{b(\underline{\xi})}, \tag{6}$$

where $\underline{\xi}$ represents the independent kinematic variables of the decay process without photons, and where

$$\frac{\mathrm{d}\Gamma_{\mathrm{Brem}}}{\mathrm{d}E_\gamma \mathrm{d}\underline{\xi}} = \frac{\mathrm{d}\Gamma_0}{\mathrm{d}\underline{\xi}} \frac{b(\underline{\xi})}{E_\gamma}. \tag{7}$$

Note that, while for two body decays b is a constant, for decays with more particles in the final state, the velocities β in (3) and thus b, depend on the variables $\underline{\xi}$.

3 MC simulation for radiative decays

While for a complete MC simulation we need the decay width for all values of E_γ, the relation in (6) is true only for soft-photon emission. However, the value of the exponent b, which is of order α, is about 0.01. Hence, while $(E_\gamma/M)^b$ is a large correction for $E_\gamma \to 0$, its value is close to 1 when $E_\gamma \to M$. Therefore, if we use the complete differential decay width at order α for the emission of a photon, $\mathrm{d}\Gamma_{\mathrm{Brem}}/\mathrm{d}E_\gamma \mathrm{d}\underline{\xi}$, instead of its approximation for low energies in (7), the decay width in (6) represents a good approximation for the entire energy spectrum.

In summary, the "recipe" for writing the amplitude for the process $i \to f\gamma$ is the following.
(1) Calculate the b factor using (2).
(2) Calculate the amplitude $M_{i\to f\gamma}$ at order α.
(3) Fix the divergence in the squared amplitude by multiplying by $(E_\gamma/M)^b$:

$$|M_{i\to f\gamma}|^2_{\text{no IR}} = |M_{i\to f\gamma}|^2 \, (E_\gamma/M)^b. \tag{8}$$

3.1 MC generators

Following the recipe outlined in the previous section we have written the MC generators for the decays $K^0 \to \pi\pi\gamma$, $K^0 \to \pi e\nu\gamma$, $K^0 \to \pi\mu\nu\gamma$, $K^0 \to \pi^+\pi^-\pi^0\gamma$, $K^\pm \to \pi\pi\gamma$, $K^\pm \to e\nu\gamma$, $K^\pm \to \mu\nu\gamma$, $K^\pm \to \pi e\nu\gamma$, $K^\pm \to \pi\mu\nu\gamma$, and $K^\pm \to \pi^\pm\pi^0\pi^0\gamma$. We obtained the amplitudes $M_{i\to f\gamma}$ at order α mainly from [7], where they are calculated using chiral perturbation theory at order p^2 and p^4. Concerning the semileptonic modes, for simplicity we have used only the p^2 expression of the $M_{i\to f\gamma}$ amplitudes. At this order, the form factors f_+ and f_- are equal to 1 and 0 respectively. In order to take into account the leading dependence on the variable $t = (p_K - p_\pi)^2$ we have multiplied the amplitude by an overall factor $(1 + \lambda_+ t/M^2_{\pi^+})$.[1] The uncertainty related to this approximation is discussed in the following.

Large-scale MC production in high energy experiments puts stringent limits on the time needed to generate one single event. This time should not exceed the time needed to track the particles inside the detector. In the KLOE MC this time is about a few milliseconds. We have used a combination of MC sampling techniques, the *acceptance-rejection* (Von Neumann) method and the *inverse transform* method [8], to reach this goal [9]. The average time for generating one event is a fraction of a millisecond.

We compared the fraction of events with a photon above an energy threshold predicted by the MC simulation, with theoretical expectations and experimental results, for several kaon decay channels. For instance, the MC prediction for the fraction of $K^0 \to \pi^+\pi^-\gamma$ in which a photon has energy greater than 20 MeV (50 MeV) is equal to 7.00×10^{-3} (2.54×10^{-3}), in agreement with the measured value $(7.10 \pm 0.22) \times 10^{-3}$ ($(2.56 \pm 0.09) \times 10^{-3}$), and that from theoretical predictions 7.01×10^{-3} (2.56×10^{-3}) for the K_S[2] [10].

Moreover, we calculated the ratio

$$R(E,\theta) = \frac{\Gamma(K_{e3\gamma}, E_\gamma > E, \theta_{e\gamma} > \theta)}{\Gamma(K_{e3}(\gamma))}, \tag{9}$$

giving the fraction of decays $k \to \pi e\nu\gamma$ with a photon with energy E_γ above E and angle between the electron and photon $\theta_{e\gamma}$ above θ, for different values of the energy E and angle θ, and we compared it to theoretical predictions and, whenever possible, with experimental values. MC calculations of R^0 for $K^0 \to \pi e\nu\gamma$ decays are shown in Table 1, while theoretical expectations from [11,12][3] are shown in the top and middle part of Table 2, respectively. Since the decay width in the denominator of (9) is inclusive of photon emission, following the treatment of [16], we divided the predictions of [11,12] by a factor $(1+\delta^e_k)$, where δ^e_k is the total electromagnetic correction extracted from [13]. Note that a different approach is used in [14], where most of the electromagnetic corrections are absorbed in $f_+(0)$ and therefore cancel out in the ratio R^0.

The results for R^0 from a recent MC simulation, described in [15], are shown in the lower part of Table 2. Experimental results for R^0 have been recently published in [16] by the KTeV Collaboration, for two values of the photon energy and angle:

$$R^0(10\,\mathrm{MeV}, 0^\circ) = (4.942 \pm 0.062) \times 10^{-2}, \tag{10}$$

$$R^0(30\,\mathrm{MeV}, 20^\circ) = (0.916 \pm 0.017) \times 10^{-2}, \tag{11}$$

[1] We used the value $\lambda_+ = 0.03$.

[2] These numbers refers only to the inner bremsstrahlung (IB) term.

[3] We quote the values from [12], obtained for $\lambda_+ = 0.03$, as in our simulation.

Table 1. Ratios $R^0 \times 10^2$ for K^0e3 decays obtained from the MC simulation with 10^7 events. The first error is statistical while the second is systematic

θ/E(MeV)	10	20	30	40
0°	4.908(7)(56)	3.252(6)(47)	2.364(5)(43)	1.784(4)(38)
10°	2.450(5)(38)	1.657(4)(36)	1.223(3)(31)	0.937(3)(28)
20°	1.864(4)(33)	1.262(4)(30)	0.933(3)(28)	0.716(3)(24)
30°	1.516(4)(30)	1.028(3)(28)	0.760(3)(25)	0.583(2)(22)
40°	1.264(4)(27)	0.859(3)(25)	0.636(3)(23)	0.488(2)(21)
50°	1.066(3)(24)	0.726(3)(23)	0.538(2)(20)	0.414(2)(19)

Table 2. Ratios $R^0 \times 10^2$ for K^0e3 decays: (top) values listed in [11]; (middle) values obtained from [12]; (bottom) values from the MC simulation in [15]. The values from [11,12] have been multiplied by $(1+\delta_k^e)^{-1}$ where $\delta_k^e = 1.04(20)\%$ from [13]

[11] θ/E(MeV)	10	20	30	40
0°	4.94	3.25	2.35	1.77
10°	2.48	1.67	1.24	0.95
20°	1.90	1.29	0.95	0.73
30°	1.55	1.05	0.78	0.60
40°	1.29	0.85	0.66	0.50
50°	1.10	0.75	0.56	0.43
[12] θ/E(MeV)	10	20	30	40
0°	–	3.27	2.37	1.78
[15] θ/E(MeV)	10	20	30	40
0°	4.93(6)	–	2.36(3)	–
20°	1.89(2)	–	0.96(1)	–

and in [17] by the NA48 Collaboration:

$$R^0(30\,\mathrm{MeV}, 20°) = (0.964 \pm 0.008^{+0.011}_{-0.009}) \times 10^{-2}. \quad (12)$$

MC calculations and theoretical expectations of $R^\pm$ for $K^\pm \to \pi e \nu \gamma$ decays are shown in Tables 3 and 4. Since the radiative correction δ_k^e for $\Gamma(K^\pm_{e3})$ is compatible with zero, $\delta_k^e = +0.06(20)\%$, it has been neglected.

In Tables 1 and 3 the first error is statistical while the second is systematic. We computed the systematic error by comparing the difference between the branching ratios calculated at order $O(p^2)$ and at order $O(p^4)$ in [7], with the variations of R we observed in the simulation when the term $(1+\lambda_+ t/M^2_{\pi^+})$ is included, or not, in the decay amplitude. In [7] the branching ratios evaluated for $E_\gamma > 30\,\mathrm{MeV}$ and $\theta_\gamma > 20°$ increase by about 6% in the $O(p^4)$ calculation, while the variations of R in the simulation are about 3%. Hence, we used the variations observed in the simulation for each value of E_γ and θ_γ as systematic errors. Moreover, we changed the value of the cut-off Λ from M_K to $M_K/2$ to check the stability of the results, resulting in negligible variations on the order of the statistical errors shown in the two tables.

For both R^0 and $R^\pm$ the absolute differences between the MC simulation and theoretical predictions are below 5×10^{-4}, and below the quoted systematic errors. Moreover, such errors are smaller than the relative errors on the measured branching ratios for kaon decays [2,3,18].

4 Conclusions

We overcome the problem of infinite probabilities in radiative processes by extending the soft-photon approximation of [6] to the whole energy range. The spectra produced with MC generators developed with this technique agree well with other theoretical calculations and with available experimental data. The systematic error could be further reduced by using full $O(p^4)$ calculations for the amplitudes, or even adding the $O(p^6)$ results recently published in [14].

Table 3. Ratios $R^\pm \times 10^2$ for $K^\pm e3$ decays obtained from the MC simulation with 10^7 events. The first error is statistical while the second is systematic

θ/E(MeV)	10	20	30	40
0°	4.223(6)(34)	2.825(5)(28)	2.069(5)(27)	1.572(4)(25)
10°	1.781(4)(18)	1.238(3)(17)	0.936(3)(17)	0.732(3)(15)
20°	1.211(3)(16)	0.854(3)(15)	0.652(3)(14)	0.515(2)(13)
30°	0.881(3)(15)	0.630(3)(14)	0.488(2)(13)	0.388(2)(12)
40°	0.659(3)(14)	0.477(2)(13)	0.373(2)(11)	0.300(2)(10)
50°	0.493(2)(12)	0.363(2)(11)	0.289(2)(11)	0.234(2)(10)

Table 4. Ratios $R^{\pm} \times 10^2$ for $K^{\pm}e3$ decays: (top) values listed in [11]; (bottom) values obtained from [12]. The radiative correction $\delta_k^e = +0.06(20)\%$ is compatible with zero and has been neglected

[11] θ/E(MeV)	10	20	30	40
0°	4.24	2.80	2.03	1.53
10°	1.78	1.22	0.91	0.71
20°	1.20	0.83	0.63	0.50
30°	0.87	0.61	0.47	0.37
40°	0.65	0.46	0.36	0.29
50°	0.48	0.35	0.27	0.22
[12] θ/E(MeV)	10	20	30	40
0°	–	2.82	2.04	1.54

We used MC sampling techniques to reduce the time needed to generate one event below 1 ms. The MC generators, routines written in Fortran, have been included in the official KLOE library and have been used for large-scale MC productions. Such routines are available upon request to the author.

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